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An elementary construction of lowering and raising operators for the trigonometric Calogero–Sutherland model

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Abstract

The quantum Calogero–Sutherland model of A_n-type (Calogero F 1971 *J. Math. Phys.* **12** 419–36, Sutherland B 1972 *Phys. Rev.* A **4** 2019–21) is completely integrable (Olshanetsky M A and Perelomov A M 1977 *Lett. Math. Phys.* **2** 7–13, Olshanetsky M A and Perelomov A M 1978 *Funct. Anal. Appl.* **12** 121–8, Olshanetsky M A and Perelomov A M 1983 *Phys. Rep.* **94** 313–404). Using this fact, we give an elementary construction of lowering and raising operators for the trigonometric case. This is similar to, but more complicated (due to the fact that the energy spectrum is not equidistant) than the construction for the rational case (Perelomov A M 1976 *ITEP Preprint* No 27).

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1. Introduction

The class of quantum systems associated with root systems was introduced in [3] (see also [4,5]) as a generalization of the Calogero–Sutherland systems [1, 2]. In these papers it was shown that the systems of A_n-type (which depend on one real parameter κ, related to the coupling constant) are quantum completely integrable systems.

For the potential $v(q) = \kappa(\kappa - 1)\sin^{-2}(q)$ and special values of this parameter, the wavefunctions correspond to the characters of groups $SU(N)$, $N = n + 1$ ($\kappa = 1$) or to zonal spherical functions ($\kappa = \frac{1}{2}, 2, 4$) (see [7]). If κ changes continuously, the wavefunctions are not related to group theory but they give an interpolation between these objects. Using appropriate variables these functions become the polynomials in n variables which are natural multidimensional generalizations of Gegenbauer polynomials (which we have for the $SU(2)$ case). The properties of such polynomials and analogous functions were considered from different points of view in many papers, of which we mention here only [8–27].

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Below we follow the approach developed in [28–31]. Using the fact that the quantum trigonometric Calogero–Sutherland system is completely integrable, we give an elementary construction of lowering and raising operators for this case. This is similar to, but more complicated (due to the fact that the energy spectrum is not equidistant) than the construction for the rational Calogero–Sutherland case [6]. The approach uses just elementary means compared with other approaches [25, 26], and may be extended to the case of arbitrary root systems.

2. The quantum CS model and GG polynomials

The quantum Calogero–Sutherland model of A_n-type [1, 2] for the trigonometric case was considered first in [2] and describes the mutual interaction of $N = n + 1$ particles moving on the circle. The coordinates of these particles are q_j, $j = 1, \ldots, N$, and the Schrödinger equation reads

$$\begin{aligned} &H\Psi^\kappa = E(\kappa)\Psi^\kappa \\ &H = -\tfrac{1}{2}\Delta + \kappa(\kappa-1)\sum_{j<k}^{N} \sin^{-2}(q_j - q_k) \qquad \Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial q_j^2}. \end{aligned} \tag{1}$$

We recall some important facts about this model following [28]. The ground-state energy and (non-normalized) wavefunction are

$$\begin{aligned} &E_0(\kappa) = 2(\rho,\rho)\kappa^2 = \tfrac{1}{6}N(N+1)(N-1)\kappa^2 \\ &\Psi_0^\kappa(q_i) = \left\{\prod_{j<k}^{N} \sin(q_j - q_k)\right\}^\kappa \end{aligned} \tag{2}$$

where ρ is the standard Weyl vector, $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$ with the sum extended over all the positive roots of A_n. The excited states depend on an n-tuple of quantum numbers $\boldsymbol{m} = (m_1, m_2, \ldots, m_n)$:

$$\begin{aligned} &H\Psi_{\boldsymbol{m}}^\kappa = E_{\boldsymbol{m}}^{(2)}(\kappa)\Psi_{\boldsymbol{m}}^\kappa \\ &E_{\boldsymbol{m}}^{(2)}(\kappa) = 2(\lambda + \kappa\rho, \lambda + \kappa\rho) \end{aligned} \tag{3}$$

where λ is the highest weight of the representation of A_n labelled by $\boldsymbol{m}$, i.e. $\lambda = \sum_{i=1}^{n} m_i\lambda_i$ and λ_i are the fundamental weights of A_n. Equation (3) has been obtained combining formulae (4.2)–(4.5) of [28]. If we substitute in (3)

$$\Psi_{\boldsymbol{m}}^\kappa(q_i) = \Psi_0^\kappa(q_i)\Phi_{\boldsymbol{m}}^\kappa(q_i) \tag{4}$$

we are led to the eigenvalue problem

$$-\Delta_2^\kappa \Phi_{\boldsymbol{m}}^\kappa = \varepsilon_{\boldsymbol{m}}^{(2)}(\kappa)\Phi_{\boldsymbol{m}}^\kappa \tag{5}$$

with

$$\Delta_2^\kappa = \frac{1}{2}\Delta + \kappa\sum_{j<k}^{N} \operatorname{cotan}(q_j - q_k)\left(\frac{\partial}{\partial q_j} - \frac{\partial}{\partial q_k}\right) \tag{6}$$

and

$$\varepsilon_{\boldsymbol{m}}^{(2)}(\kappa) = E_{\boldsymbol{m}}^{(2)}(\kappa) - E_0(\kappa) = 2(\lambda, \lambda + 2\kappa\rho). \tag{7}$$

Introducing the inverse Cartan matrix

$$A_{jk}^{-1} = (\lambda_j, \lambda_k) = \min(j,k) - \frac{jk}{N} \tag{8}$$

it is possible to give a more explicit expression for $\varepsilon_{\boldsymbol{m}}(\kappa)$:

$$\begin{aligned}\varepsilon_{\boldsymbol{m}}^{(2)}(\kappa) &= 2\sum_{j,k=1}^{n} A_{jk}^{-1} m_j m_k + 4\kappa \sum_{j,k=1}^{n} A_{jk}^{-1} m_j \\ &= \frac{2}{N}\sum_{k=1}^{n} k(N-k)m_k^2 + \frac{4}{N}\sum_{l<k}^{n} l(N-k)m_l m_k + 2\kappa \sum_{k=1}^{n} k(N-k)m_k. \end{aligned} \tag{9}$$

In order to find the eigenfunctions $\Phi_{\boldsymbol{m}}^{\kappa}(q_i)$, it is convenient to introduce a set of barycentric coordinates

$$q'_j = q_j - q \qquad q = \frac{1}{N}\sum_{j=1}^{N} q_j \tag{10}$$

and change variables to the following set of elementary symmetric functions of $x_j = \mathrm{e}^{2\mathrm{i}q'_j}$:

$$\begin{aligned} z_1 &= \sum_{j=1}^{N} x_j \\ z_2 &= \sum_{j<k}^{N} x_j x_k \\ z_3 &= \sum_{j<k<l}^{N} x_j x_k x_l \\ &\vdots \\ z_N &= x_1 x_2 \ldots x_N. \end{aligned} \tag{11}$$

We will fix the centre of mass at the origin of q-coordinates. Then, $q'_j = q_j$, $z_N = 1$ and the only independent variables are $z_1, z_2, \ldots, z_n$. The Δ_2^{κ} operator becomes

$$\Delta_2^{\kappa} = \sum_{j,k=1}^{N} g_{jk}(z_i)\partial_{z_j}\partial_{z_k} + \sum_{j=1}^{N} a_j(z_i)\partial_{z_j} \tag{12}$$

where

$$\begin{aligned} g_{jk}(z_i) &= 2A_{jk}^{-1} z_j z_k + \text{ lower-order terms} \\ a_j(z_i) &= 2(1+N\kappa)A_{jj}^{-1} z_j = \frac{2}{N}(1+N\kappa)j(N-j)z_j. \end{aligned} \tag{13}$$

As a consequence of the simple form of Δ_2^{κ}, the $\Phi_{\boldsymbol{m}}^{\kappa}$ are polynomials

$$\Phi_{\boldsymbol{m}}^{\kappa}(z_i) = P_{\boldsymbol{m}}^{\kappa}(z_i) = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} + \cdots \tag{14}$$

which, for the A_1-case, are standard Gegenbauer polynomials, and for A_n, constitute a natural generalization of Gegenbauer polynomials for n variables. Some relevant properties of these polynomials as well as specific examples can be found in [5, 24, 28–31].

As an illustration, we give the form of Δ_2^{κ} and its eigenvalues for A_2 and A_3:

- For A_2 the inverse Cartan matrix is

$$A^{-1} = \tfrac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{15}$$

We write $\boldsymbol{m} = (m, n)$ and find

$$\begin{aligned} -\Delta_2^{\kappa} &= \tfrac{4}{3}\{(z_1^2 - 3z_2)\partial_{z_1}^2 + (z_2^2 - 3z_1)\partial_{z_2}^2 + (z_1 z_2 - 9)\partial_{z_1}\partial_{z_2} + (3\kappa+1)(z_1\partial_{z_1} + z_2\partial_{z_2})\} \\ \varepsilon_{m,n}^{(2)}(\kappa) &= \tfrac{4}{3}\{m^2 + n^2 + mn + 3\kappa(m+n)\}. \end{aligned} \tag{16}$$

- For A_3 we obtain

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix} \tag{17}$$

and putting $\boldsymbol{m} = (m, l, n)$, we get

$$\begin{aligned} -\Delta_2^\kappa = \tfrac{1}{2}\{&(3z_1^2 - 8z_2)\partial_{z_1}^2 + (3z_3^2 - 8z_2)\partial_{z_3}^2 + 4(z_2^2 - 2z_1z_3 - 4)\partial_{z_2}^2 \\ &+ 4(z_1z_2 - 6z_3)\partial_{z_1}\partial_{z_2} + 4(z_2z_3 - 6z_1)\partial_{z_2}\partial_{z_3} + 2(z_1z_3 - 16)\partial_{z_1}\partial_{z_3} \\ &+ (4\kappa + 1)(3z_1\partial_{z_1} + 3z_3\partial_{z_3} + 4z_2\partial_{z_2})\} \\ \varepsilon_{m,l,n}^{(2)}(\kappa) = \tfrac{1}{2}\{&3m^2 + 3n^2 + 4l^2 + 4ml + 4nl + 2mn + 4\kappa(3m + 3n + 4l)\}. \end{aligned} \tag{18}$$

3. Complete set of quantum integrals of motion

The system under consideration is completely integrable. This means that there are n commuting operators including the Hamiltonian. They may be constructed as follows. Let us introduce the operator-valued matrix of order N

$$L_{jk} = p_j\delta_{jk} - \mathrm{i}g(1 - \delta_{jk})\sin^{-1}(q_j - q_k) \qquad p_j = -\mathrm{i}\frac{\partial}{\partial q_j} \qquad g^2 = \kappa(\kappa - 1) \tag{19}$$

and let $\tilde{\Delta}_j^\kappa$ be the sum of all principal minors of order j. It is easy to see that these operators are well defined (there is no problem with ordering operators p_j in them) and that $\tilde{\Delta}_2^\kappa$ coincides with the Hamiltonian in (1). The main statement [3–5] is that these operators commute with one another:

$$[\tilde{\Delta}_j^\kappa, \tilde{\Delta}_k^\kappa] = 0 \tag{20}$$

and therefore the wavefunctions are eigenfunctions of all of them:

$$-\tilde{\Delta}_j^\kappa \Psi_{\boldsymbol{m}}^\kappa = E_{\boldsymbol{m}}^{(j)}(\kappa)\Psi_{\boldsymbol{m}}^\kappa. \tag{21}$$

The explicit form of these operators is as follows:

$$\tilde{\Delta}_j^\kappa = (-\mathrm{i})^j \sum_{l=0}^{[j/2]} g^{2l} v^{(2l)}\partial^{(j-2l)} \tag{22}$$

with

$$v^{(2l)}\partial^{(j-2l)} = \sum_C v_{i_1,i_2}\cdots v_{i_{2l-1},i_{2l}} \frac{\partial}{\partial q_{i_{2l+1}}}\cdots\frac{\partial}{\partial q_{i_j}} \qquad v_{i,j} = \sin^{-2}(q_i - q_j) \tag{23}$$

and C is the set of all non-equivalent combinations of non-repeated indices between 1 and N.

The former hierarchy of commuting operators can be transformed in another one which includes the Δ_2^κ introduced in (6). The substitution of (4) in (21) leads to the equation

$$-((\Psi_0^\kappa)^{-1}\tilde{\Delta}_j^\kappa\Psi_0^\kappa)\cdot\Phi_{\boldsymbol{m}}^\kappa = E_{\boldsymbol{m}}^{(j)}\Phi_{\boldsymbol{m}}^\kappa \tag{24}$$

and, in particular, in the case $\boldsymbol{m} = 0$ to

$$-((\Psi_0^\kappa)^{-1}\tilde{\Delta}_j^\kappa\Psi_0^\kappa)\cdot\mathbf{1} = E_0^{(j)} \tag{25}$$

where $\mathbf{1}$ is the function identically equal to one. It is therefore convenient to define the new set of operators as

$$\Delta_j^\kappa = :(\Psi_0^\kappa)^{-1}\tilde{\Delta}_j^\kappa\Psi_0^\kappa: \tag{26}$$

where the meaning of the normal-ordering operator is the following: all derivatives are displaced to the right and, among the new terms which arise as a result of this displacement, those which are purely multiplicative give a constant which we subtract. In terms of these new operators, equation (21) takes the form

$$-\Delta_j^\kappa \Phi_m^\kappa = \varepsilon_m^{(j)}(\kappa)\Phi_m^\kappa \qquad \varepsilon_m^{(j)}(\kappa) = E_m^{(j)}(\kappa) - E_0^{(j)}(\kappa). \tag{27}$$

The construction of Δ_k^κ involves the following replacement in (22):

$$\partial_j \to \partial_j + \kappa A_j \qquad A_j = \kappa^{-1}(\Psi_0^\kappa)^{-1}(\partial_j \Psi_0^\kappa) = \sum_{k\neq j} \operatorname{cotan}(q_j - q_k) \qquad j = 1, \ldots, N \tag{28}$$

After this 'gauge transformation' has been done and the normal reordering has been applied, we obtain the following results:

$$\begin{aligned}
\Delta_2^\kappa &= (-\mathrm{i})^2 \sum_C \{\partial_j\partial_k + \kappa A_j\partial_k\} \\
\Delta_3^\kappa &= (-\mathrm{i})^3 \sum_C \{\partial_j\partial_k\partial_l + \kappa A_j\partial_k\partial_l + \kappa^2[(\partial_j A_k) + A_j A_k]\partial_l\} \\
\Delta_4^\kappa &= (-\mathrm{i})^4 \sum_C \{\partial_j\partial_k\partial_l\partial_m + \kappa A_j\partial_k\partial_l\partial_m + \kappa^2[(\partial_j A_k) + A_j A_k]\partial_l\partial_m \\
&\quad + \kappa^3[(\partial_j A_k)A_l + A_j A_k A_l]\partial_m\}
\end{aligned} \tag{29}$$

and so on. The sums are over all non-equivalent combinations of non-repeated indices between 1 and N (note that $(\partial_j A_k) = (\partial_k A_j)$). It is easy to check that the first operator of the preceding list coincides with that of (6), as it should. The other operators can be put in more explicit form in each concrete case. For instance, for A_2,

$$\begin{aligned}
-\mathrm{i}\Delta_3^\kappa = \partial_1\partial_2\partial_3 &+ \kappa\{[\operatorname{cotan}(q_1 - q_2) + \operatorname{cotan}(q_1 - q_3)]\partial_2\partial_3 \\
&+ [\operatorname{cotan}(q_2 - q_1) + \operatorname{cotan}(q_2 - q_3)]\partial_1\partial_3 \\
&+ [\operatorname{cotan}(q_3 - q_1) + \operatorname{cotan}(q_3 - q_2)]\partial_1\partial_2\} \\
&+ 2\kappa^2\{[1 + \operatorname{cotan}(q_3 - q_1)\operatorname{cotan}(q_3 - q_2)]\partial_3 \\
&+ [1 + \operatorname{cotan}(q_2 - q_1)\operatorname{cotan}(q_2 - q_3)]\partial_2 \\
&+ [1 + \operatorname{cotan}(q_1 - q_2)\operatorname{cotan}(q_1 - q_3)]\partial_1\}.
\end{aligned} \tag{30}$$

After the change of variables (11), we get

$$\begin{aligned}
\Delta_3^\kappa = \left(\tfrac{2}{3}\right)^3\{(2z_1^3 - 9z_1z_2 + 27)\partial_{z_1}^3 &+ (3z_1^2z_2 - 18z_2^2 + 27z_1)\partial_{z_1}^2\partial_{z_2} \\
&- (3z_1z_2^2 - 18z_1^2 + 27z_2)\partial_{z_1}\partial_{z_2}^2 - (2z_2^3 - 9z_1z_2 + 27)\partial_{z_2}^3 \\
&+ 3(3\kappa + 2)[(z_1^2 - 3z_2)\partial_{z_1}^2 - (z_2^2 - 3z_1)\partial_{z_2}^2] \\
&+ (3\kappa + 2)(3\kappa + 1)(z_1\partial_{z_1} - z_2\partial_{z_2})\}.
\end{aligned} \tag{31}$$

4. Raising and lowering operators

In this section, we will show how to build raising and lowering operators for the Gegenbauer polynomials associated with A_n. After explaining the general treatment, we will give the explicit form of these operators for A_2- and A_3-cases. Our approach relies on combining the characteristic polynomial for the Lax matrix (19) with the recurrence relations satisfied by the Gegenbauer polynomials [28]. The normal-ordered characteristic polynomial for the Lax matrix takes the form

$$D(t) = \det(t\boldsymbol{I} - L) = t^N + \sum_{j=2}^{N} (-1)^j \tilde{\Delta}_j^\kappa t^{N-j}. \tag{32}$$

As a result of (27), the generalized Gegenbauer polynomials are eigenfunctions of $D(t)$. If we apply normal ordering and use the shifted operator

$$\Delta(t) = {:}D(t){:} - \sum_{j=2}^{N} (-1)^j E_0^{(j)} t^{N-j} \tag{33}$$

the eigenvalue equations take the form

$$\Delta(t) P_{\boldsymbol{m}}^\kappa = \prod_{j=1}^{N} \left(t - l_{\boldsymbol{m}}^{(j)}\right) P_{\boldsymbol{m}}^\kappa \tag{34}$$

where $l_{\boldsymbol{m}}^{(j)}$ are the components of the N-dimensional vector $\boldsymbol{l_m} = 2(\lambda + \kappa\rho)$:

$$l_{\boldsymbol{m}}^{(j)} = \frac{2}{N} \left\{ \sum_{k=1}^{n} (N-k) m_k - N \sum_{k=0}^{j-1} m_k + \frac{1}{2} N(N+1-2j)\kappa \right\}. \tag{35}$$

The easiest way to check this equation is by means of analytical continuation of the coordinates, $q_j \to \mathrm{i}q_j$, to the asymptotic region $q_j \gg q_k$ if $j > k$, in which only the diagonal part of L and the leading term of $P_{\boldsymbol{m}}^\kappa$ survive.

On the other hand, the recurrence relations among Gegenbauer polynomials are deformations of the Clebsch–Gordan series for $SU(N)$, specifically

$$z_r P_{\boldsymbol{m}}^\kappa = \sum_{i_1 < i_2 < \cdots < i_r}^{N} a_{i_1, i_2, \ldots, i_r}(\kappa) P_{\boldsymbol{m}+\boldsymbol{\mu}_{i_1}+\cdots+\boldsymbol{\mu}_{i_r}}^\kappa \qquad r = 1, 2, \ldots, n \tag{36}$$

or, alternatively,

$$z_{N-r} P_{\boldsymbol{m}}^\kappa = \sum_{i_1 < i_2 < \cdots < i_r}^{N} b_{i_1, i_2, \ldots, i_r}(\kappa) P_{\boldsymbol{m}-\boldsymbol{\mu}_{i_1}-\cdots-\boldsymbol{\mu}_{i_r}}^\kappa \qquad r = 1, 2, \ldots, n. \tag{37}$$

Here $b_{i_1,\ldots,i_r}(\kappa) = a_{i_{r+1},\ldots,i_N}$ if $\{i_1, \ldots, i_N\} = \{1, \ldots, N\}$ and $\boldsymbol{\mu}_i$, with i going from 1 to N, are n-dimensional vectors whose components are

$$\boldsymbol{\mu}_i = (\delta_{k,i} - \delta_{k,i-1}) \qquad k = 1, 2, \ldots, n. \tag{38}$$

Using the explicit form (38) in (35), we find

$$l_{\boldsymbol{m}\pm\boldsymbol{\mu}_{i_1}\pm\cdots\pm\boldsymbol{\mu}_{i_r}}^{(j)} = l_{\boldsymbol{m}}^{(j)} \mp \frac{2r}{N} \pm 2\delta_{i_1}^j \pm \cdots \pm 2\delta_{i_r}^j. \tag{39}$$

Bearing in mind (34), this implies that $\Delta(l_{\boldsymbol{m}}^{(j)} - 2r/N)$ is zero when applied to all terms of (36) which do not involve $\boldsymbol{\mu}_j$ and, similarly, $\Delta(l_{\boldsymbol{m}}^{(j)} + 2r/N)$ vanishes when acting on the terms of (37) not including $-\boldsymbol{\mu}_j$. Thus,

$$\begin{aligned}
&\Delta\left(l_{\boldsymbol{m}}^{(i_1)} - \frac{2r}{N}\right) \Delta\left(l_{\boldsymbol{m}}^{(i_2)} - \frac{2r}{N}\right) \cdots \Delta\left(l_{\boldsymbol{m}}^{(i_r)} - \frac{2r}{N}\right) z_r P_{\boldsymbol{m}}^\kappa \propto P_{\boldsymbol{m}+\boldsymbol{\mu}_{i_1}+\cdots+\boldsymbol{\mu}_{i_r}}^\kappa \\
&\Delta\left(l_{\boldsymbol{m}}^{(i_1)} + \frac{2r}{N}\right) \Delta\left(l_{\boldsymbol{m}}^{(i_2)} + \frac{2r}{N}\right) \cdots \Delta\left(l_{\boldsymbol{m}}^{(i_r)} + \frac{2r}{N}\right) z_{N-r} P_{\boldsymbol{m}}^\kappa \propto P_{\boldsymbol{m}-\boldsymbol{\mu}_{i_1}-\cdots-\boldsymbol{\mu}_{i_r}}^\kappa
\end{aligned} \tag{40}$$

and are therefore these products which give the desired raising and lowering operators, in this case annihilating $P_{\boldsymbol{m}}^\kappa$ and creating $P_{\boldsymbol{m}\pm\boldsymbol{\mu}_{i_1}\pm\cdots\pm\boldsymbol{\mu}_{i_r}}^\kappa$.

Let us now concentrate on the A_2-case, for which $\boldsymbol{m} = (m, n)$ and the $l^{(j)}_{m,n}$ are

$$\begin{aligned} l^{(1)}_{m,n} &= \tfrac{2}{3}(2m + n + 3\kappa) \\ l^{(2)}_{m,n} &= \tfrac{2}{3}(-m + n) \\ l^{(3)}_{m,n} &= \tfrac{2}{3}(-m - 2n - 3\kappa). \end{aligned} \tag{41}$$

The explicit form of the recurrence relations [28] is

$$\begin{aligned} z_1 P^\kappa_{m,n} &= P^\kappa_{m+1,n} + a_{m,n}(\kappa) P^\kappa_{m,n-1} + c_m(\kappa) P^\kappa_{m-1,n+1} \\ z_2 P^\kappa_{m,n} &= P^\kappa_{m,n+1} + a_{n,m}(\kappa) P^\kappa_{m-1,n} + c_n(\kappa) P^\kappa_{m+1,n-1} \end{aligned} \tag{42}$$

where

$$\begin{aligned} a_{m,n}(\kappa) &= \frac{n(m+n+\kappa)(n-1+2\kappa)(m+n-1+3\kappa)}{(n+\kappa)(n-1+\kappa)(m+n+2\kappa)(m+n-1+2\kappa)} \\ c_m(\kappa) &= \frac{m(m-1+2\kappa)}{(m+\kappa)(m-1+\kappa)}. \end{aligned} \tag{43}$$

Using the general construction explained above, and with the notation

$$S_{a,b} P^\kappa_{m,n} = \sigma^{m,n}_{a,b} P^\kappa_{m+a,n+b} \tag{44}$$

we find the following raising and lowering operators and corresponding proportionality factors:

$$\begin{aligned} S_{1,0} &= \Delta\big(l^{(1)}_{m,n} - \tfrac{2}{3}\big) z_1 & \sigma^{m,n}_{1,0} &= -h_{m,n}(\kappa) \\ S_{-1,1} &= \Delta\big(l^{(2)}_{m,n} - \tfrac{2}{3}\big) z_1 & \sigma^{m,n}_{-1,1} &= k_{m,n}(\kappa) c_m(\kappa) \\ S_{0,-1} &= \Delta\big(l^{(3)}_{m,n} - \tfrac{2}{3}\big) z_1 & \sigma^{m,n}_{0,-1} &= -h_{n,m}(\kappa) a_{m,n}(\kappa) \\ S_{-1,0} &= \Delta\big(l^{(1)}_{m,n} + \tfrac{2}{3}\big) z_2 & \sigma^{m,n}_{-1,0} &= h_{m,n}(\kappa) a_{n,m}(\kappa) \\ S_{1,-1} &= \Delta\big(l^{(2)}_{m,n} + \tfrac{2}{3}\big) z_2 & \sigma^{m,n}_{1,-1} &= -k_{m,n}(\kappa) c_n(\kappa) \\ S_{0,1} &= \Delta\big(l^{(3)}_{m,n} + \tfrac{2}{3}\big) z_2 & \sigma^{m,n}_{0,1} &= h_{n,m}(\kappa) \end{aligned} \tag{45}$$

where the new coefficients $h_{m,n}(\kappa)$ and $k_{m,n}(\kappa)$ are

$$\begin{aligned} h_{m,n}(\kappa) &= 2^3(m + n + 2\kappa)(m + \kappa) \\ k_{m,n}(\kappa) &= 2^3(m + \kappa)(n + \kappa). \end{aligned} \tag{46}$$

Let us now move to the A_3-case, for which we will write $\boldsymbol{m} = (m, l, n)$. The $l^{(j)}_{m,l,n}$ are

$$\begin{aligned} l^{(1)}_{m,l,n} &= \tfrac{1}{2}(3m + 2l + n + 6\kappa) \\ l^{(2)}_{m,l,n} &= \tfrac{1}{2}(-m + 2l + n + 2\kappa) \\ l^{(3)}_{m,l,n} &= \tfrac{1}{2}(-m - 2l + n - 2\kappa) \\ l^{(4)}_{m,l,n} &= \tfrac{1}{2}(-m - 2l - 3n - 6\kappa). \end{aligned} \tag{47}$$

The recurrence relations have the form

$$\begin{aligned} z_1 P^\kappa_{m,l,n} &= P^\kappa_{m+1,l,n} + c_m(\kappa) P_{m-1,l+1,n} + a_{m,l}(\kappa) P^\kappa_{m,l-1,n+1} + d_{m,l,n}(\kappa) P^\kappa_{m,l,n-1} \\ z_2 P^\kappa_{m,l,n} &= P^\kappa_{m,l+1,n} + c_l(\kappa) P_{m+1,l-1,n+1} + a_{l,m}(\kappa) P^\kappa_{m+1,l,n-1} + a_{l,n}(\kappa) P^\kappa_{m-1,l,n+1} \\ &\quad + f_{m,l,n}(\kappa) P^\kappa_{m-1,l+1,n-1} + g_{m,l,n}(\kappa) P^\kappa_{m,l-1,n} \\ z_3 P^\kappa_{m,l,n} &= P^\kappa_{m,l,n+1} + c_n(\kappa) P_{m,l+1,n-1} + a_{n,l}(\kappa) P^\kappa_{m+1,l-1,n} + d_{n,l,m}(\kappa) P^\kappa_{m-1,l,n} \end{aligned} \tag{48}$$

where the coefficients $a_{p,q}(\kappa)$ and $c_p(\kappa)$ are as in (43), and

$$\begin{aligned}
d_{m,l,n}(\kappa) &= \frac{n(l+n+\kappa)(n-1+2\kappa)(m+l+n+2\kappa)(l+n-1+3\kappa)}{(n+\kappa)(n-1+\kappa)(l+n+2\kappa)(l+n-1+2\kappa)(m+l+n+3\kappa)} \\
&\quad \times \frac{(m+l+n-1+4\kappa)}{(m+l+n-1+3\kappa)} \\
f_{m,l,n}(\kappa) &= \frac{mn(m-1+2\kappa)(n-1+2\kappa)(m+l+n+2\kappa)(m+l+n-1+4\kappa)}{(m+\kappa)(n+\kappa)(m-1+\kappa)(n-1+\kappa)(m+l+n+3\kappa)(m+l+n-1+3\kappa)} \\
g_{m,l,n}(\kappa) &= \frac{l(m+l+\kappa)(l+n+\kappa)(l-1+2\kappa)(m+l+n+2\kappa)(m+l-1+3\kappa)}{(l+\kappa)(l-1+\kappa)(m+l+2\kappa)(m+l-1+2\kappa)(l+n+2\kappa)(l+n-1+2\kappa)} \\
&\quad \times \frac{(l+n-1+3\kappa)(m+l+n-1+4\kappa)}{(m+l+n+3\kappa)(m+l+n-1+3\kappa)}.
\end{aligned} \tag{49}$$

With the notation (44), the raising and lowering operators are as follows:

$$\begin{aligned}
S_{1,0,0} &= \Delta\big(l^{(1)}_{m,l,n} - \tfrac{1}{2}\big)z_1 & \sigma^{m,l,n}_{1,0,0} &= -q_{m,l,n}(\kappa) \\
S_{-1,1,0} &= \Delta\big(l^{(2)}_{m,l,n} - \tfrac{1}{2}\big)z_1 & \sigma^{m,l,n}_{-1,1,0} &= r_{m,l,n}(\kappa)c_m(\kappa) \\
S_{0,-1,1} &= \Delta\big(l^{(3)}_{m,l,n} - \tfrac{1}{2}\big)z_1 & \sigma^{m,l,n}_{0,-1,1} &= -r_{n,l,m}(\kappa)a_{m,l}(\kappa) \\
S_{0,0,-1} &= \Delta\big(l^{(4)}_{m,l,n} - \tfrac{1}{2}\big)z_1 & \sigma^{m,l,n}_{0,0,-1} &= q_{n,l,m}(\kappa)d_{m,l,n}(\kappa) \\
S_{0,0,1} &= \Delta\big(l^{(4)}_{m,l,n} + \tfrac{1}{2}\big)z_3 & \sigma^{m,l,n}_{0,0,1} &= -q_{n,l,m}(\kappa) \\
S_{0,1,-1} &= \Delta\big(l^{(3)}_{m,l,n} + \tfrac{1}{2}\big)z_3 & \sigma^{m,l,n}_{0,1,-1} &= r_{n,l,m}(\kappa)c_n(\kappa) \\
S_{1,-1,0} &= \Delta\big(l^{(2)}_{m,l,n} + \tfrac{1}{2}\big)z_3 & \sigma^{m,l,n}_{1,-1,0} &= -r_{m,l,n}(\kappa)a_{n,l}(\kappa) \\
S_{-1,0,0} &= \Delta\big(l^{(1)}_{m,l,n} + \tfrac{1}{2}\big)z_3 & \sigma^{m,l,n}_{-1,0,0} &= q_{m,l,n}(\kappa)d_{n,l,m}(\kappa)
\end{aligned} \tag{50}$$

$$\begin{aligned}
S_{0,1,0} &= \Delta\big(l^{(1)}_{m,l,n} - 1\big)\Delta\big(l^{(2)}_{m,l,n} - 1\big)z_2 & \sigma^{m,l,n}_{0,1,0} &= -p_{m,l,n}(\kappa) \\
S_{1,-1,1} &= \Delta\big(l^{(1)}_{m,l,n} - 1\big)\Delta\big(l^{(3)}_{m,l,n} - 1\big)z_2 & \sigma^{m,l,n}_{1,-1,1} &= t_{m,l,n}(\kappa)c_l(\kappa) \\
S_{1,0,-1} &= \Delta\big(l^{(1)}_{m,l,n} - 1\big)\Delta\big(l^{(4)}_{m,l,n} - 1\big)z_2 & \sigma^{m,l,n}_{1,0,-1} &= -w_{m,l,n}(\kappa)a_{l,m}(\kappa) \\
S_{-1,0,1} &= \Delta\big(l^{(2)}_{m,l,n} - 1\big)\Delta\big(l^{(3)}_{m,l,n} - 1\big)z_2 & \sigma^{m,l,n}_{1,-1,1} &= -x_{m,l,n}(\kappa)a_{l,n}(\kappa) \\
S_{-1,1,-1} &= \Delta\big(l^{(2)}_{m,l,n} - 1\big)\Delta\big(l^{(4)}_{m,l,n} - 1\big)z_2 & \sigma^{m,l,n}_{-1,1,-1} &= t_{n,l,m}(\kappa)f_{m,l,n}(\kappa) \\
S_{0,-1,0} &= \Delta\big(l^{(3)}_{m,l,n} - 1\big)\Delta\big(l^{(4)}_{m,l,n} - 1\big)z_2 & \sigma^{m,l,n}_{0,-1,0} &= -p_{n,l,m}(\kappa)g_{m,l,n}(\kappa)
\end{aligned} \tag{51}$$

where

$$\begin{aligned}
q_{m,l,n}(\kappa) &= 2^4(m+\kappa)(m+l+2\kappa)(m+l+n+3\kappa) \\
r_{m,l,n}(\kappa) &= 2^4(m+\kappa)(l+\kappa)(l+n+2\kappa) \\
p_{m,l,n}(\kappa) &= 2^8(l+\kappa)(m+1+\kappa)(m-1+\kappa)(m+l+2\kappa)(l+n+2\kappa)(m+l+n+3\kappa) \\
t_{m,l,n}(\kappa) &= 2^8(m+\kappa)(l+\kappa)(n+\kappa)(l+m+1+2\kappa)(m+l-1+2\kappa)(m+l+n+3\kappa) \\
w_{m,l,n}(\kappa) &= 2^8(m+\kappa)(n+\kappa)(m+l+2\kappa)(l+n+2\kappa) \\
&\quad \times (m+l+n+1+3\kappa)(m+l+n-1+3\kappa) \\
x_{m,l,n}(\kappa) &= 2^8(m+\kappa)(n+\kappa)(l+1+\kappa)(l-1+\kappa)(m+l+2\kappa)(l+n+2\kappa).
\end{aligned} \tag{52}$$

5. Conclusions

In this letter, we have described a procedure for building raising and lowering operators for the system of generalized Gegenbauer polynomials associated with the root system of A_n. This procedure has been applied to obtain the step operators for the cases of A_2 and A_3. In the latter case, we have also written for the first time the explicit form of the recurrence relations among the polynomials. Also, we give in the appendix the exact expression of some of the lowest-order polynomials for A_3.

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Appendix. Explicit expressions for Gegenbauer polynomials for the A_3-case up to total degree four

We provide a list of some Gegenbauer polynomials for the A_3-case which extends that given in [29] for the A_2-case.

$$P^{\kappa}_{1,0,0} = z_1$$
$$P^{\kappa}_{0,1,0} = z_2$$
$$P^{\kappa}_{2,0,0} = z_1^2 - \frac{2}{1+\kappa} z_2$$
$$P^{\kappa}_{0,2,0} = z_2^2 - \frac{2}{1+\kappa} z_1 z_3 - \frac{2(\kappa-1)}{(1+\kappa)(1+2\kappa)}$$
$$P^{\kappa}_{1,1,0} = z_1 z_2 - \frac{3}{1+2\kappa} z_3$$
$$P^{\kappa}_{1,0,1} = z_1 z_3 - \frac{4}{1+3\kappa}$$
$$P^{\kappa}_{3,0,0} = z_1^3 - \frac{6}{2+\kappa} z_1 z_2 + \frac{6}{(1+\kappa)(2+\kappa)} z_3$$
$$P^{\kappa}_{0,3,0} = z_2^3 - \frac{6}{2+\kappa} z_1 z_2 z_3 + \frac{6}{(1+\kappa)(2+\kappa)}(z_1^2 + z_3^2) - \frac{3(2+\kappa+\kappa^2)}{(1+\kappa)^2(2+\kappa)} z_2$$
$$P^{\kappa}_{2,1,0} = z_1^2 z_2 - \frac{2}{1+\kappa} z_2^2 - \frac{1+3\kappa}{(1+\kappa)^2} z_1 z_3 + \frac{4}{(1+\kappa)^2}$$
$$P^{\kappa}_{2,0,1} = z_1^2 z_3 - \frac{2}{1+\kappa} z_2 z_3 - \frac{2(1+4\kappa)}{(1+\kappa)(2+3\kappa)} z_1$$
$$P^{\kappa}_{1,2,0} = z_1 z_2^2 - \frac{2}{1+\kappa} z_1^2 z_3 - \frac{1+3\kappa}{(1+\kappa)^2} z_2 z_3 - \frac{\kappa-5}{(1+\kappa)^2} z_1$$
$$P^{\kappa}_{1,1,1} = z_1 z_2 z_3 - \frac{3}{1+2\kappa}(z_1^2 + z_3^2) - \frac{8(\kappa-1)}{(1+2\kappa)(2+3\kappa)} z_2$$
$$P^{\kappa}_{4,0,0} = z_1^4 - \frac{12}{3+\kappa} z_1^2 z_2 + \frac{12}{(2+\kappa)(3+\kappa)} z_2^2 + \frac{24}{(2+\kappa)(3+\kappa)} z_1 z_3 - \frac{24}{(1+\kappa)(2+\kappa)(3+\kappa)}$$

$$P^{\kappa}_{3,1,0} = z_1^3 z_2 - \frac{6}{2+\kappa} z_1 z_2^2 - \frac{3(2+3\kappa)}{(2+\kappa)(3+2\kappa)} z_1^2 z_3 + \frac{30}{(2+\kappa)(3+2\kappa)} z_2 z_3 + \frac{6(1+4\kappa)}{(1+\kappa)(2+\kappa)(3+2\kappa)} z_1$$

$$P^{\kappa}_{3,0,1} = z_1^3 z_3 - \frac{6}{2+\kappa} z_1 z_2 z_3 - \frac{2(1+2\kappa)}{(1+\kappa)(2+\kappa)} z_1^2 + \frac{6}{(1+\kappa)(2+\kappa)} z_3^2 + \frac{4(1+2\kappa)}{(1+\kappa)^2(2+\kappa)} z_2$$

$$P^{\kappa}_{2,0,2} = z_1^2 z_3^2 - \frac{2}{(1+\kappa)}(z_2 z_3^2 + z_1^2 z_2) + \frac{4}{(1+\kappa)^2} z_2^2 - \frac{8\kappa(1+2\kappa)}{3(1+\kappa)^3} z_1 z_3 - \frac{8(3+\kappa-4\kappa^2)}{3(1+\kappa)^3(2+3\kappa)}$$

$$P^{\kappa}_{2,2,0} = z_1^2 z_2^2 - \frac{2}{1+\kappa}(z_1^3 z_3 + z_2^3) + \frac{12(1-\kappa)}{(1+\kappa)(3+2\kappa)} z_1 z_2 z_3 + \frac{2(3+8\kappa-\kappa^2)}{(3+2\kappa)(1+\kappa)^2} z_1^2 - \frac{9(1-\kappa)}{(3+2\kappa)(1+\kappa)^2} z_3^2 + \frac{2(3+7\kappa+10\kappa^2)}{(3+2\kappa)(1+\kappa)^3} z_2$$

$$P^{\kappa}_{2,1,1} = z_1^2 z_2 z_3 - \frac{3}{1+2\kappa} z_1^3 - \frac{2}{1+\kappa} z_2^2 z_3 - \frac{1+3\kappa}{(1+\kappa)^2} z_1 z_3^2 + \frac{2(12+25\kappa+7\kappa^2-8\kappa^3)}{3(1+2\kappa)(1+\kappa)^3} z_1 z_2 + \frac{2(-1+5\kappa+8\kappa^2)}{(1+2\kappa)(1+\kappa)^3} z_3$$

$$P^{\kappa}_{1,3,0} = z_1 z_2^3 - \frac{6}{2+\kappa} z_1^2 z_2 z_3 + \frac{30}{(2+\kappa)(3+2\kappa)} z_1 z_3^2 - \frac{3(2+3\kappa)}{(2+\kappa)(3+2\kappa)} z_2^2 z_3 + \frac{6}{2+3\kappa+\kappa^2} z_1^3 - \frac{6(2-3\kappa+\kappa^2)}{(1+\kappa)(2+\kappa)(3+2\kappa)} z_1 z_2 - \frac{3(10+13\kappa-3\kappa^2)}{(2+\kappa)(3+2\kappa)(1+\kappa)^2} z_3$$

$$P^{\kappa}_{1,2,1} = z_1 z_2^2 z_3 - \frac{2}{1+\kappa} z_1^2 z_3^2 - \frac{1+3\kappa}{(1+\kappa)^2}(z_1^2 z_2 + z_2 z_3^2) + \frac{4\kappa(1-\kappa)}{3(1+\kappa)^3} z_2^2 + \frac{30+73\kappa+44\kappa^2-3\kappa^3}{3(1+\kappa)^4} z_1 z_3 - \frac{4(6+7\kappa-\kappa^2)}{3(1+\kappa)^4}$$

$$P^{\kappa}_{0,4,0} = z_2^4 - \frac{12}{3+\kappa} z_1 z_2^2 z_3 + \frac{12}{(2+\kappa)(3+\kappa)} z_1^2 z_3^2 + \frac{24}{(2+\kappa)(3+\kappa)}(z_1^2 z_2 + z_2 z_3^2) - \frac{12(6+3\kappa+\kappa^2)}{(2+\kappa)(3+\kappa)(3+2\kappa)} z_2^2 - \frac{24(6-\kappa)}{(2+\kappa)(3+\kappa)(3+2\kappa)} z_1 z_3 + \frac{6(18+\kappa+\kappa^2)}{(1+\kappa)(2+\kappa)(3+\kappa)(3+2\kappa)}.$$

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